Stress-Strain Equation Plane Stress and Plane Strain Equations

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In matrix form, Hooke's law for isotropic materials can be written as

$$\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{bmatrix} = \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \gamma_{23}\\ \gamma_{13}\\ \gamma_{12}\end{bmatrix} = \frac{1}{E}\begin{bmatrix}1 & -\nu & -\nu & 0 & 0 & 0\\ -\nu & 1 & -\nu & 0 & 0 & 0\\ -\nu & -\nu & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 2+2\nu & 0 & 0\\ 0 & 0 & 0 & 0 & 2+2\nu & 0\\ 0 & 0 & 0 & 0 & 0 & 2+2\nu\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix}$$

where $\gamma_{ij} = 2\varepsilon_{ij}$ is the **engineering shear strain**. The inverse relation may be written as

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{bmatrix}$$

§5.5.2. Stress-To-Strain Relations

To get stresses if the strains are given, the most expedient method is to invert the matrix equation (5.16). This gives

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} \hat{E}\,(1-\nu) & \hat{E}\,\nu & \hat{E}\,\nu & 0 & 0 & 0 \\ \hat{E}\,\nu & \hat{E}\,(1-\nu) & \hat{E}\,\nu & 0 & 0 & 0 \\ \hat{E}\,\nu & \hat{E}\,\nu & \hat{E}\,(1-\nu) & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}. \tag{5.17}$$

Here $\hat{E}$ is an "effective" modulus modified by Poisson's ratio:

$$\hat{E} = \frac{E}{(1-2\nu)(1+\nu)} \tag{5.18}$$

linear isotropic elasticity.

Material	Poisson's ratio
Isotropic upper limit [1]	0.5
Rubber [6]	0.48- ~0.5
Indium [11]	0.45
Gold [4]	0.42
Lead [4]	0.44
Copper [7]	0.37
Aluminum [4]	0.34
Copper [4]	0.35
Polystyrene [6]	0.34
Brass [1]	0.33
Ice [8]	0.33
Polystyrene foam [6]	0.3
Stainless Steel [7]	0.30
Steel [1]	0.29
Tungsten [4]	0.30
Tungsten	0.28
Fused quartz [9]	0.17
Boron [12]	0.08
Beryllium [4]	0.03
Re-entrant foam [10]	-0.7
Isotropic lower limit [1]	-1

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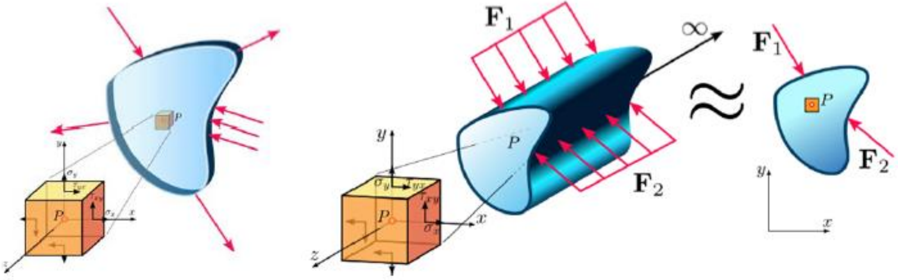
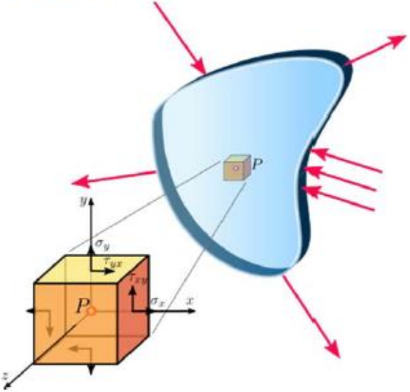
Plane Stress and Plane Strain

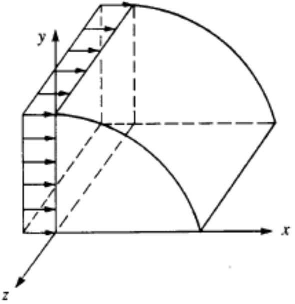


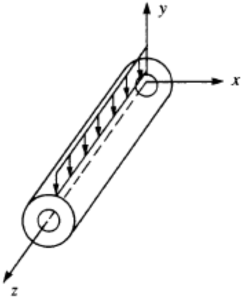
The two-dimensional element is extremely important for:

(1) ***Plane stress analysis***, which includes problems such as plates with holes, fillets, or other changes in geometry that are loaded in their plane resulting in local stress concentrations.



(2) ***Plane strain analysis***, which includes problems such as a long underground box culvert subjected to a uniform load acting constantly over its length or a long cylindrical control rod subjected to a load that remains constant over the rod length (or depth).





Plane Stress

Plane stress is defined to be ***a state of stress in which the normal stress and the shear stresses directed perpendicular to the plane are assumed to be zero***.

That is, the normal stress σ_z and the shear stresses τ_{xz} and τ_{yz} are assumed to be zero.

Generally, members that are thin (those with a small *z* dimension compared to the in-plane *x* and *y* dimensions) and whose loads act only in the *x*-*y* plane can be considered to be under plane stress.

Plane Strain

Plane strain is defined to be ***a state of strain in which the strain normal to the x-y plane ε_z and the shear strains γ_{xz} and γ_{yz} are assumed to be zero.***

The assumptions of plane strain are realistic for long bodies (say, in the *z* direction) with constant cross-sectional area subjected to loads that act only in the *x* and/or *y* directions and do not vary in the *z* direction.

Two-Dimensional State of Stress and Strain

For **plane stress,** the stresses σ_z, τ_{xz}, and τ_{yz} are assumed to be zero. The stress-strain relationship is:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad [D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix}$$

is called the ***stress-strain matrix*** (or the ***constitutive matrix***), *E* is the modulus of elasticity, and ν is Poisson's ratio.

Two-Dimensional State of Stress and Strain

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$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad [D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix}$$

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The limits of Poisson's ratio for isotropic solids possess fundamental significance. Shape is preserved at the lower limit of $v = -1$ (applicable for both 3D and 2D). Volume is preserved at the upper limit $v = 1/2$ (for 3D) while area is preserved at the upper limit of $v = 1$ (for 2D). It is now of interest, though not in a practical sense, to present the bounds of Poisson's ratio under 1D, 2D and 3D analyses as

$$\begin{array}{ll} v = 0; & d = 1 \\ -1 \leq v \leq 1; & d = 2 \\ -1 \leq v \leq 1/2; & d = 3 \end{array} \tag{3.2.17}$$

whereby d = 1, 2, 3 refer to the number of dimensions. Of course the so-called "bound" for d = 1 is not a bound but this has been included for the sake of completeness. Alternatively, the bounds for 2D and 3D can be combined to give

$$\begin{array}{ll} v = 0; & d = 1 \\ -1 \leq v \leq \frac{1}{d-1}; & d = 2, 3 \end{array} . \tag{3.2.18}$$

In addition to the Poisson's ratio bounds based on 3D analysis, it is possible to obtain the Poisson's ratio bounds for 2D. The upper bound of Poisson's ratio for 2D case can be performed either on the basis of plane strain or plane stress. In addition to $\sigma_{ij} = -p$; $(i = j)$ and $\sigma_{ij} = 0$; $(i \neq j)$ for hydrostatic pressure, the plane strain condition requires that $e_{33} = 0$. Of course the plane strain condition also implies $e_{23} = e_{31} = 0$ but these have no effect on our calculation. From Hooke's Law in 2D,

$$e_{11} = e_{22} \propto \frac{p}{E}(v - 1). \tag{3.2.13}$$

Since $e_{11} = e_{22} \leq 0$ due to the hydrostatic pressure and $E \geq 0$, we have $v - 1 \leq 0$ or

$$v \leq 1. \tag{3.2.14}$$

As before, the imposition of $e_{11} = e_{22} \leq 0$ arising from hydrostatic pressure and $E \geq 0$ leads to Eq. (3.2.14). Whether by plane strain ($e_{33} = 0$) or by plane stress ($\sigma_{33} = 0$), the strain energy for 2D analysis is common

$$U \propto \frac{p^2}{E}(1 - \nu) \tag{3.2.16}$$

because $\sigma_{33} e_{33} = 0$ for both cases under hydrostatic pressure. On the basis of $U \geq 0$ and $E \geq 0$, Eq. (3.2.14) is recovered for 2D analysis. Practically, the assumption of plane strain is more plausible since it is not possible to impose plane stress condition under hydrostatic pressure. The lower limit for the Poisson's ratio in 2D analysis is similar to that of 3D, because the condition of simple shear has only one stress component $\sigma_{23} = \tau$ regardless of 3D or 2D analyses.

Plane Strain Equations