Stress-Strain Equation

Plane Stress and Plane Strain Equations

The stiffness and compliance tensors [edit]

For hyperelastic materials, the stress and strain of a linear elastic material are such that one can be derived from a stored energy potential function of the other (also called a strain energy density function). Therefore, we can define an elastic material to be one which satisfies

$$\boldsymbol{\sigma} = \frac{\partial w(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} \quad \text{or} \quad \sigma_{ij} = \frac{\partial w}{\partial \varepsilon_{ij}}$$

where w is the strain energy density function.

If the material, in addition to being elastic, also has a linear stress-strain relation then we can write

$$\boldsymbol{\sigma} = \mathsf{C} : \boldsymbol{\varepsilon} \quad \text{or} \quad \sigma_{ij} = C_{ijkl}\, \varepsilon_{kl}$$

The quantity C is called the **stiffness tensor** or the **elasticity tensor**.

Therefore, the strain energy density function has the form (this form is called a **quadratic form**)

$$w(\boldsymbol{\varepsilon}) = \frac{1}{2}\,\boldsymbol{\varepsilon} : \mathbf{C} : \boldsymbol{\varepsilon} = \frac{1}{2}\,C_{ijkl}\,\varepsilon_{ij}\,\varepsilon_{kl}$$

Clearly, the elasticity tensor has 81 components (think of a 9×9 matrix because the stresses and strains have nine components each). However, the symmetries of the stress tensor implies that

$$C_{ijkl} = C_{jikl}$$

This reduces the number independent components of C_{ijkl} to 54 (6 components for the ij term and 3 each for the k, l terms.

Similarly, using the symmetry of the strain tensor we can show that

$$C_{ijkl} = C_{ijlk}$$

These are called the **minor symmetries** of the elasticity tensor and we are then left with only 36 components that are independent.

Since the strain energy function should not change when we interchange ij and kl in the quadratic form, we must have

$$C_{ijkl} = C_{klij}$$

This reduces the number of independent constants to 21 (think of a symmetric 6×6 matrix). These are called the **major symmetries** of the stiffness tensor.

The inverse relation between the strain and the stress can be determined by taking the inverse of stress-strain relation to get

$$\boldsymbol{\varepsilon} = \mathbf{S} : \boldsymbol{\sigma} \quad \text{or} \quad \varepsilon_{ij} = S_{ijkl}\, \sigma_{kl}$$

where $\mathbf{S}$ is the **compliance tensor**. The compliance tensor also has 21 components and the same symmetries as the stiffness tensor.

Voigt notation

To express the general stress-strain relation for a linear elastic material in terms of matrices we use what is called the *Voigt notation*.
In this notation, the stress and strain are expressed as column vectors and the elasticity tensor is expressed as a symmetric matrix as shown below.

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{31}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112}\\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212}\\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3331} & C_{3312}\\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2331} & C_{2312}\\ C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3131} & C_{3112}\\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1231} & C_{1212}\end{bmatrix} \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\,\varepsilon_{23}\\ 2\,\varepsilon_{31}\\ 2\,\varepsilon_{12}\end{bmatrix}$$

or

$$\sigma = \mathbf{C}\,\varepsilon$$

The inverse relation is

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\,\varepsilon_{23} \\ 2\,\varepsilon_{31} \\ 2\,\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1133} & 2\,S_{1123} & 2\,S_{1131} & 2\,S_{1112} \\ S_{2211} & S_{2222} & S_{2233} & 2\,S_{2223} & 2\,S_{2231} & 2\,S_{2212} \\ S_{3311} & S_{3322} & S_{3333} & 2\,S_{3323} & 2\,S_{3331} & 2\,S_{3312} \\ 2\,S_{2311} & 2\,S_{2322} & 2\,S_{2333} & 4\,S_{2323} & 4\,S_{2331} & 4\,S_{2312} \\ 2\,S_{3111} & 2\,S_{3122} & 2\,S_{3133} & 4\,S_{3123} & 4\,S_{3131} & 4\,S_{3112} \\ 2\,S_{1211} & 2\,S_{1222} & 2\,S_{1233} & 4\,S_{1223} & 4\,S_{1231} & 4\,S_{1212} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix}$$

or

$$\varepsilon = \mathbf{S}\,\sigma$$

We can show that

$$\mathbf{S} = \mathbf{C}^{-1}$$

Isotropic materials [edit]

We have already seen the matrix form of the stress-strain equation for isotropic linear elastic materials. In this case the stiffness tensor has only two independent components because every plane is a plane of elastic symmetry. In direct tensor notation

$$\mathsf{C} = \lambda\, \mathit{1} \otimes \mathit{1} + 2\,\mu\, \mathsf{I}$$

where λ and μ are the elastic constants that we defined before, $\mathit{1}$ is the second-order identity tensor, and I is the symmetric fourth-order identity tensor. In index notation

$$C_{ijkl} = \lambda\, \delta_{ij}\, \delta_{kl} + 2\,\mu\, \frac{1}{2}(\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk})$$

One could alternatively express this equation in terms of:

- the Young's modulus () and the Poisson's ratio () or
- in terms of the bulk modulus () and the shear modulus () or
- any other combination of two independent elastic parameters.

In Voigt notation the expression for the stress-strain law for isotropic materials can be written as

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & (C_{11}-C_{12})/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (C_{11}-C_{12})/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (C_{11}-C_{12})/2 \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\,\varepsilon_{23} \\ 2\,\varepsilon_{31} \\ 2\,\varepsilon_{12} \end{bmatrix}$$

where

$$C_{11} = \frac{E\,(1-\nu)}{(1+\nu)(1-2\nu)}\ ;\ C_{12} = \frac{E\,\nu}{(1+\nu)(1-2\nu)}\ ;\ (C_{11}-C_{12})/2 = \frac{E}{2(1+\nu)} = \mu\,.$$

The Voigt form of the strain-stress relation can be written as

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\,\varepsilon_{23} \\ 2\,\varepsilon_{31} \\ 2\,\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{12} & S_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\,(S_{11}-S_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\,(S_{11}-S_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\,(S_{11}-S_{12}) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix}$$

where

$$S_{11} = \frac{1}{E}\,;\;\; S_{12} = -\frac{\nu}{E}\,;\;\; 2\,(S_{11}-S_{12}) = \frac{2(1+\nu)}{E} = \frac{1}{\mu}\,.$$

The relations between various moduli are shown in the table below:

	μ, ν	ν, λ	μ, λ	K, λ	μ, E	μ, K	ν, E	ν, K	K, E
λ	$\frac{2\,\mu\,\nu}{1-2\nu}$	-	-	-	$\frac{\mu\,(E-2\mu)}{3\mu-E}$	$K-\frac{2}{3}\,\mu$	$\frac{E\,\nu}{(1+\nu)(1-2\nu)}$	$\frac{3\,K\,\nu}{1+\nu}$	$\frac{3\,K\,(3K-E)}{9K-E}$
μ	-	$\frac{\lambda\,(1-2\nu)}{2\nu}$	-	$\frac{3}{2}(K-\lambda)$	-	-	$\frac{E}{2(1+\nu)}$	$\frac{3K(1-2\nu)}{2(1+\nu)}$	$\frac{3KE}{9K-E}$
ν	-	-	$\frac{\lambda}{2(\lambda+\mu)}$	$\frac{\lambda}{3K-\lambda}$	$\frac{E-2\mu}{2\mu}$	$\frac{3K-2\mu}{2(3K+\mu)}$	-	-	$\frac{3K-E}{6K}$
E	$2\mu(1+\nu)$	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$	$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\frac{9K(K-\lambda)}{3K-\lambda}$	-	$\frac{9K\mu}{3K+\mu}$	-	$3K(1-2\nu)$	-
K	$\frac{2\mu(1+\nu)}{3(1-2\nu)}$	$\frac{\lambda(1+\nu)}{3\nu}$	$\lambda+\frac{2}{3}\,\mu$	-	$\frac{\mu E}{3(3\mu-E)}$	-	$\frac{E}{3(1-2\nu)}$	-	-

In matrix form, Hooke's law for isotropic materials can be written as

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2+2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2+2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2+2\nu \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}$$

where $\gamma_{ij} = 2\varepsilon_{ij}$ is the **engineering shear strain**. The inverse relation may be written as

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix}$$

§5.5.2. Stress-To-Strain Relations

To get stresses if the strains are given, the most expedient method is to invert the matrix equation (5.16). This gives

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} \hat{E}\,(1-\nu) & \hat{E}\,\nu & \hat{E}\,\nu & 0 & 0 & 0 \\ \hat{E}\,\nu & \hat{E}\,(1-\nu) & \hat{E}\,\nu & 0 & 0 & 0 \\ \hat{E}\,\nu & \hat{E}\,\nu & \hat{E}\,(1-\nu) & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}. \tag{5.17}$$

Here $\hat{E}$ is an "effective" modulus modified by Poisson's ratio:

$$\hat{E} = \frac{E}{(1-2\nu)(1+\nu)} \tag{5.18}$$

linear isotropic elasticity.

Material	Poisson's ratio
Isotropic upper limit [1]	0.5
Rubber [6]	0.48- ~0.5
Indium [11]	0.45
Gold [4]	0.42
Lead [4]	0.44
Copper [7]	0.37
Aluminum [4]	0.34
Copper [4]	0.35
Polystyrene [6]	0.34
Brass [1]	0.33
Ice [8]	0.33
Polystyrene foam [6]	0.3
Stainless Steel [7]	0.30
Steel [1]	0.29
Tungsten [4]	0.30
Tungsten	0.28
Fused quartz [9]	0.17
Boron [12]	0.08
Beryllium [4]	0.03
Re-entrant foam [10]	-0.7
Isotropic lower limit [1]	-1

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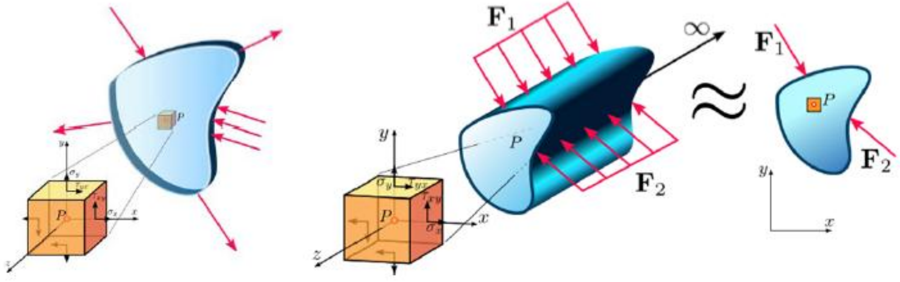
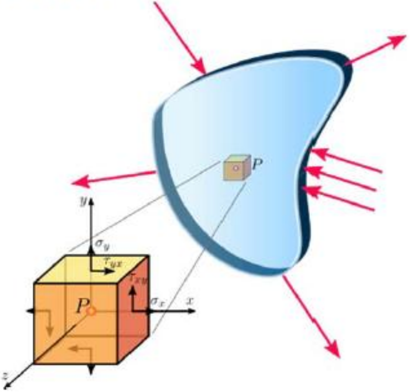
Plane Stress and Plane Strain

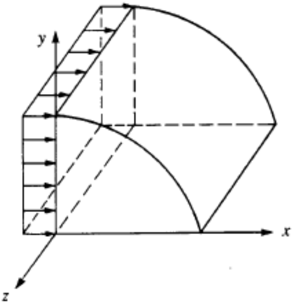


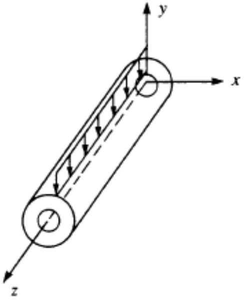
The two-dimensional element is extremely important for:

(1) ***Plane stress analysis***, which includes problems such as plates with holes, fillets, or other changes in geometry that are loaded in their plane resulting in local stress concentrations.



(2) ***Plane strain analysis***, which includes problems such as a long underground box culvert subjected to a uniform load acting constantly over its length or a long cylindrical control rod subjected to a load that remains constant over the rod length (or depth).





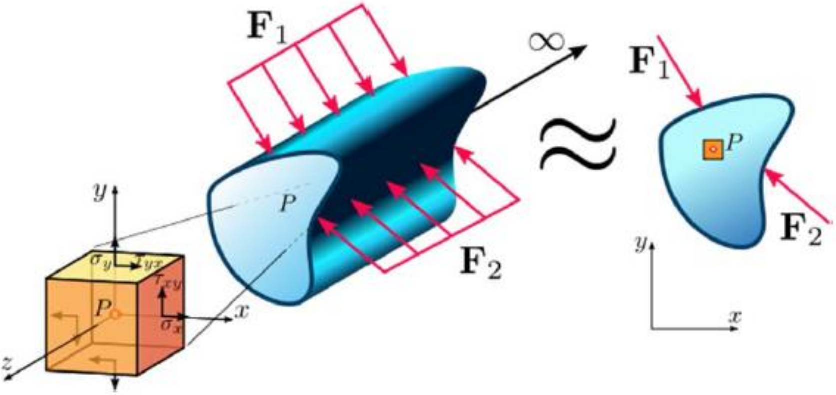
Plane Stress

Plane stress is defined to be ***a state of stress in which the normal stress and the shear stresses directed perpendicular to the plane are assumed to be zero***.

That is, the normal stress σ_z and the shear stresses τ_{xz} and τ_{yz} are assumed to be zero.

Generally, members that are thin (those with a small *z* dimension compared to the in-plane *x* and *y* dimensions) and whose loads act only in the *x*-*y* plane can be considered to be under plane stress.

Plane Strain Equations



Plane Strain

Plane strain is defined to be ***a state of strain in which the strain normal to the x-y plane ε_z and the shear strains γ_{xz} and γ_{yz} are assumed to be zero.***

The assumptions of plane strain are realistic for long bodies (say, in the *z* direction) with constant cross-sectional area subjected to loads that act only in the *x* and/or *y* directions and do not vary in the *z* direction.

Two-Dimensional State of Stress and Strain

For **plane stress,** the stresses σ_z, τ_{xz}, and τ_{yz} are assumed to be zero. The stress-strain relationship is:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad [D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix}$$

is called the ***stress-strain matrix*** (or the ***constitutive matrix***), E is the modulus of elasticity, and ν is Poisson's ratio.

Two-Dimensional State of Stress and Strain

For **plane strain**, the strains ε_z, γ_{xz}, and γ_{yz} are assumed to be zero. The stress-strain relationship is:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad [D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix}$$

is called the ***stress-strain matrix*** (or the ***constitutive matrix***), E is the modulus of elasticity, and ν is Poisson's ratio.

The limits of Poisson's ratio for isotropic solids possess fundamental significance. Shape is preserved at the lower limit of $v = -1$ (applicable for both 3D and 2D). Volume is preserved at the upper limit $v = 1/2$ (for 3D) while area is preserved at the upper limit of $v = 1$ (for 2D). It is now of interest, though not in a practical sense, to present the bounds of Poisson's ratio under 1D, 2D and 3D analyses as

$$\begin{aligned} & v = 0; && d = 1 \\ & -1 \leq v \leq 1; && d = 2 \\ & -1 \leq v \leq 1/2; && d = 3 \end{aligned} \qquad (3.2.17)$$

whereby $d = 1, 2, 3$ refer to the number of dimensions. Of course the so-called "bound" for $d = 1$ is not a bound but this has been included for the sake of completeness. Alternatively, the bounds for 2D and 3D can be combined to give

$$\begin{aligned} & v = 0; && d = 1 \\ & -1 \leq v \leq \tfrac{1}{d-1}; && d = 2, 3 \end{aligned} \qquad (3.2.18)$$

In addition to the Poisson's ratio bounds based on 3D analysis, it is possible to obtain the Poisson's ratio bounds for 2D. The upper bound of Poisson's ratio for 2D case can be performed either on the basis of plane strain or plane stress. In addition to $\sigma_{ij} = -p$; $(i = j)$ and $\sigma_{ij} = 0$; $(i \neq j)$ for hydrostatic pressure, the plane strain condition requires that $e_{33} = 0$. Of course the plane strain condition also implies $e_{23} = e_{31} = 0$ but these have no effect on our calculation. From Hooke's Law in 2D,

$$e_{11} = e_{22} \propto \frac{p}{E}(v - 1). \tag{3.2.13}$$

Since $e_{11} = e_{22} \leq 0$ due to the hydrostatic pressure and $E \geq 0$, we have $v - 1 \leq 0$ or

$$v \leq 1. \tag{3.2.14}$$

As before, the imposition of $e_{11} = e_{22} \leq 0$ arising from hydrostatic pressure and $E \geq 0$ leads to Eq. (3.2.14). Whether by plane strain ($e_{33} = 0$) or by plane stress ($\sigma_{33} = 0$), the strain energy for 2D analysis is common

$$U \propto \frac{p^2}{E}(1 - \nu) \tag{3.2.16}$$

because $\sigma_{33} e_{33} = 0$ for both cases under hydrostatic pressure. On the basis of $U \geq 0$ and $E \geq 0$, Eq. (3.2.14) is recovered for 2D analysis. Practically, the assumption of plane strain is more plausible since it is not possible to impose plane stress condition under hydrostatic pressure. The lower limit for the Poisson's ratio in 2D analysis is similar to that of 3D, because the condition of simple shear has only one stress component $\sigma_{23} = \tau$ regardless of 3D or 2D analyses.